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A statistical theory of time-dependent fracture for cementitious materials subjected to cyclic loading

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Direct tensile fatigue data of both normal and lightweight concrete have been analysed with a statistical time-dependent fracture theory. Although originally developed for single-phase materials, the theory gives quite good agreement with experimental results for two reasons. One is that the inhomogeneous properties of concrete have been included in the statistically determined Weibull parameters, so that it can be treated as a pseudo-homogeneous material. The other is that at low stress ratios, less than 0.2, the basic crack growth law, whether it is due to environmental effect and/or fatigue-induced damage, can be represented by a single equation: $\dot{a} = AK^n$ where (A, n) are equivalent parameters for the two crack mechanisms. For high stress ratios the crack velocity, $\dot{a}$, cannot be represented by a single equation and the statistical fracture theory breaks down under these situations.

1. Introduction

Strength degradation due to environment-assisted slow crack growth has been widely accepted in the explanation of time-dependent fracture of brittle materials under constant sustained loadings and of the stress rate effect on strength [1–6]. A pre-existing flaw in a brittle material will extend to its critical length under a sustained load according to a particular slow crack growth law (i.e. Equation 5) which leads to eventual failure. In high stress rate experiments in the laboratory, the strength of a brittle material is determined by its pre-existing flaw distribution because there is not enough time for slow crack growth to occur. But the situation is different in low stress rate testing. In this case the strength is controlled by a flaw distribution whose average length is bigger than that of the pre-existing flaw distribution, because slow crack growth can now take place. Therefore, a stress rate effect on strength is shown. Based on a single-crack consideration a time-dependent fracture theory has been developed [2–6], and extensively used in the time-dependent fracture analysis of glasses and ceramics.

It has been shown that cementitious materials such as cement paste, mortar and concrete are also subjected to time-dependent fracture [7–10]. But as distinct from glasses and ceramics, cementitious materials, especially concrete, are very inhomogeneous in structure. Nadeau *et al.* [7], as well as Baldie and Pratt [11] have found that the slow crack growth parameters (i.e. A and n of Equation 5) of a hardened cement paste are not constant as they are supposed to be.This phenomenon is seldom observed in ceramic materials. Microscopy study of fracture in hardened cement pastes shows a tortuous crack path around unhydrated cement grains which effectively bridges the crack faces. Crack branching also occurs. We have found [12] that changes in the slow crack growth parameters can be contributed by the bridging fibres over cracks in a short-fibre reinforced cement, or by the tough second-phase particles impeding cracks in a two-phase material. Using the equivalent crack growth parameters (A, n) obtained by disregarding the structural inhomogeneity and the resistance of the second-phase particles to crack growth, lifetimes of a two-phase material under constant loads can be predicted from the results of constant stress rates. This conclusion indicates that the time-dependent fracture theory proposed for brittle isotropic and homogeneous materials can be applied to mortar and concrete as far as lifetime prediction is concerned. Indeed from a statistical viewpoint it is not unreasonable to assume these cementitious materials to be macroscopically isotropic and homogeneous.

The single crack approach to cyclic fatigue in brittle materials has been performed by Evans and Fuller [13] based on the environment-assisted slow crack growth analysis. Such an approach is inadequate for otherwise smooth specimens subjected to non-uniform stress fields. To overcome this problem we have developed a statistical time-dependent fracture theory [14] by considering a time-dependent flaw distribution, and successfully applied it to a soda glass and a polycrystalline alumina. It is proven that the lifetime or number of cycles to fracture of glass and alumina under cyclic loading can be predicted in terms of the slow crack growth parameters obtained under sustained loading conditions. Therefore, crack growth mechanisms are identical in cyclic and in sustained load situations. Cyclic fatigue-induced damage is insignificant in these materials. Recently, Tait [15] has shown that in dry mortar the equivalent crack velocity

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under cyclic loading can be accurately predicted from crack velocity data obtained for constant sustained loading. Moreover, the cyclic crack velocity is independent of fatigue frequency. These results seem to suggest that crack growth is basically a time-dependent phenomenon with identical mechanisms for both cyclic and sustained loads. Provided the equivalent parameters (A, n) are obtained for the inhomogeneous mortar, it can be treated as if it were a single-phase homogeneous material. Thus, our statistical time-dependent fracture theory is expected to apply to mortar. Concrete is more complex in structure and we wish to investigate if and under what conditions our statistical fracture theory can apply to predict lifetimes of these concrete materials. In particular we wish to determine if fatigue-induced damage exists in concrete.

2. Theoretical basis

The most accepted statistical theory used in the analysis of fracture strength of brittle materials is Weibull's weakest link theory [16, 17]. In a two-parameter form the theory is given by

$$F(\sigma) = 1 - \exp\left[-V\left[\frac{\sigma}{\sigma_v}\right]^m\right] \quad (1)$$

with m and σ_v to be determined. From Griffith's theory it is known that the strength of a brittle material can be related to the fracture of the most critical flaw in the material. Hence, the weakest link theory could be devolped from flaw statistics, and the strength distribution of Equation 1 could be related to the flaw size distribution $q(a)$ in the material.

Hunt and McCartney [18] have shown that for volume distributed flaws the failure probability at stress σ is given by

$$F(\sigma) = 1 - \exp\left[-V\int_{a(\sigma)}^{\infty} q(a)\,\mathrm{d}a\right] \quad (2)$$

under the assumption that the material is subjected to uniaxial tension. $a(\sigma)$ is given by Griffith's theory

$$a(\sigma) = \left(\frac{K_{IC}}{Y\sigma}\right)^2 \quad (3)$$

where Y is a geometry factor. If $q(\sigma)$ is described by a Pareto distribution

$$q(a) = \begin{cases} \varrho_f \dfrac{m}{2a_0}\left(\dfrac{a_0}{a}\right)^{m+2}; & a_0 \leqslant a \\ 0; & a < a_0 \end{cases} \quad (4)$$

where a_0 is the smallest flaw size. Equation 1 can be obtained by integrating Equation 2.

To study the time-dependent fracture, we assume that all pre-existing flaws in a brittle material will propagate in accordance with a slow crack growth law. The most common equation used to describe the environment-assisted slow crack growth in brittle materials is [19, 20]

$$\frac{\mathrm{d}a}{\mathrm{d}t} = AK^n = A(\sigma Y a^{1/2})^n \quad (5)$$

where $\mathrm{d}a/\mathrm{d}t$ is the crack growth velocity, and (A, n) are numerical constants depending on the particular material–environment system.

The growth of flaws within the stressed volume, V, alters the flaw size distribution from $q(\sigma)$ [$= q(a, 0)$] to $q(a, t)$. Thus, according to Equation 2, the failure probability at time t is given by

$$F(\sigma, t) = 1 - \exp\left[-V\int_{a(\sigma)}^{\infty} q(a, t)\,\mathrm{d}a\right] \quad (6)$$

It has been proven that [14]

$$\int_{a(\sigma)}^{\infty} q(a, t)\,\mathrm{d}a = \int_{a_f(\sigma,t)}^{\infty} q(a)\,\mathrm{d}a \quad (7)$$

where $a_f(\sigma, t)$ is related to $a(\sigma)$ by

$$\int_{a_f(\sigma,t)}^{a(\sigma)} a^{-n/2}\mathrm{d}a = \int_0^t A(\sigma Y)^n\,\mathrm{d}t \quad (8)$$

Thus, the integral in Equation 6 can be evaluated.

We first consider the stress-rate effect in uniaxial tension and assume $\sigma(t) = \dot{\sigma}t$ where $\dot{\sigma}$ = constant. From Equations 6 to 8, we have

$$\begin{aligned} F(\dot{\sigma}t, t) &= 1 - \exp\left\{-\left[\frac{\dot{\sigma}t}{\sigma_T}\right]^m \times \left[1 + \frac{(n-2)}{2(n+1)} AY^2K_{Ic}^{n-2}(\dot{\sigma}t)^2 t\right]^{m/(n-2)}\right\} \\ &\approx 1 - \exp\left\{-\left[\frac{\dot{\sigma}t}{\sigma_T}\right]^m \times \left[\frac{(n-2)}{2(n+1)} AY^2K_{Ic}^{n-2}(\dot{\sigma}t)^2 t\right]^{m/(n-2)}\right\} \end{aligned} \quad (9)$$

The approximation holds when the slow crack growth effect becomes signifcant. The equation can also be expressed in the relation of strength against stress rate, i.e.

$$\log \sigma_f = \frac{1}{n+1}\log\dot{\sigma} + \frac{1}{n+1}\log\left[C_T\left(\ln\frac{1}{1-F}\right)^{(n-2)/m}\right] \quad (10)$$

where C_T is a constant given by

$$C_T = \frac{2(n+1)\sigma_T^{n-2}}{(n-2)AY^2K_{Ic}^{n-2}} \quad (11)$$

We now consider the time-dependent fracture under a cyclic stress situation. There are generally two types of cyclic stresses. One is $\sigma(t) = \sigma_c \sin\omega t$ in rotation, the other is $\sigma(t) = \bar{\sigma} + \sigma_c \sin\omega t$ in either tension or bending. If cyclic fatigue-induced damage can be neglected in cementitious materials, as shown by Tait [15], the crack growth due to cyclic loading can be obtained from Equation 8 by considering the time-dependent applied stress $\sigma(t)$ and flaw statistics. Consider specimens of volume V under a tensile cyclic stress. We have

$$\begin{aligned} \sigma(t) &= \bar{\sigma} + \sigma_c \sin\omega t \\ &= \bar{\sigma}(1 + \zeta\sin\omega t) \end{aligned} \quad (12)$$

where $0 < \zeta < 1$. From Equations 6, 8 and 12, it can be shown that

$$F(\sigma, t) = 1 - \exp\left\{-\left(\frac{\sigma_{max}}{\sigma_T}\right)^m\right.$$

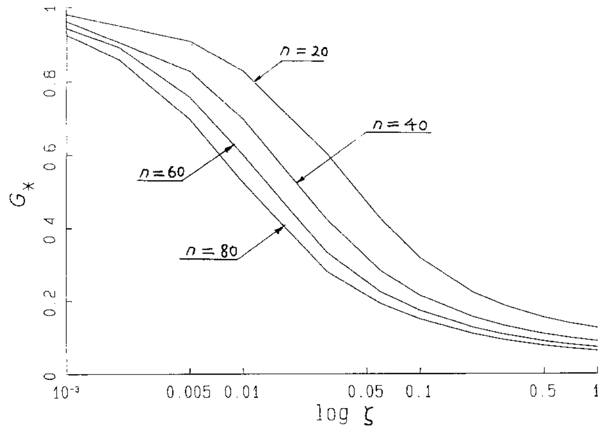

Figure 1 Influence of n and ζ on G^*.

$$\times\left(1+\frac{(n-2)}{2}AY^2K_{\mathrm{Ic}}^{n-2}\sigma_{\max}^2 t\left[\frac{G(n,\zeta)}{(1+\zeta)^n}\right]\right)^{m/(n-2)}\Bigg\}$$

$$\approx 1-\exp\left\{-\left(\frac{\sigma_{\max}}{\sigma_{\mathrm{T}}}\right)^m\right.$$

$$\left.\times\left(\frac{(n-2)}{2}AY^2K_{\mathrm{Ic}}^{n-2}\sigma_{\max}^2 t\left[\frac{G(n,\zeta)}{(1+\zeta)^n}\right]\right)^{m/(n-2)}\right\} \quad (13)$$

Here

$$\sigma_{\max} = \bar{\sigma}+\sigma_{\mathrm{c}} \quad (14)$$

$$G(n,\zeta) = \frac{1}{t}\int_0^t (1+\zeta\sin\omega t)^n \mathrm{d}t \quad (15)$$

Again, the simplified relation in Equation 13 holds when the slow crack growth effect becomes important. Equation 13 can be further simplified, so that

$$\log\ln\frac{1}{1-F} = \frac{m}{n-2}\log N + \frac{m}{n-2}\log\left\{\frac{\sigma_{\max}^n}{C_{\mathrm{T}}f}\left[\frac{G(n,\zeta)}{(1+\zeta)^n}\right](1+n)\right\} \quad (16)$$

where N ($=ft$, f being the frequency) is fatigue life in number of cycles. Let

$$G_*(n,\zeta) = \frac{G(n,\zeta)}{(1+\zeta)^n} \quad (17)$$

and this function is shown in Fig. 1 for $0 < \zeta < 1$ and $20 < n < 80$.

If n and C_{T} are determined (Weibull modulus m is determined from high stress rate inert strength tests) from constant stress rate tests using Equation 10, then the fatigue life under a cyclic stress is predictable from Equation 16.

Similarly, the lifetime relations due to a constant stress rate and a cyclic stress can be obtained for rectangular specimens of volume-distributed flaws under pure bending. Thus we have

$$\log\sigma_{\mathrm{f}} = \frac{1}{n+1}\log\dot{\sigma} + \frac{1}{n+1}\times\log\left\{C_{\mathrm{B}}\left(\ln\frac{1}{1-F}\right)^{(n-2)/m}\times\left[\frac{mn+n-2}{(m+1)(n-2)}\right]^{(n-2)/m}\right\} \quad (18)$$

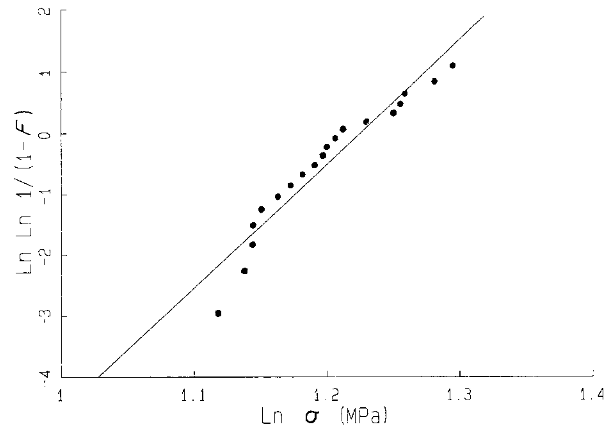

Figure 2 Tensile strength of plain concrete replotted from Saito and Imai [21].

for a constant stress rate condition. Here C_{B} is a constant and given by Equation 11 with $\sigma_{\mathrm{T}} = \sigma_{\mathrm{B}}$. Also

$$\log\ln\frac{1}{1-F} = \frac{m}{n-2}\log N + \frac{m}{n-2}\log\left\{\frac{\sigma_{\max}^n}{C_{\mathrm{B}}f}\left[\frac{G(n,\zeta)}{(1+\zeta)^n}\right](n+1)\times\left[\frac{mn+n-2}{(n-2)(m+1)}\right]^{(n-2)/m}\right\} \quad (19)$$

for a cyclic stress condition.

3. Comparison of theory with direct tensile fatigue data of concrete

Saito and Imai [21] performed a series of direct tensile fatigue experiments on plain concrete. The concrete was made using crushed stone with maximum size of 20 mm, river sand and ordinary Portland cement. The ratio of the minimum dimension of the tensile specimens to the maximum size of coarse aggregate was 3.5. The water–cement ratio, sand–cement ratio and cement content were 0.54, 0.47 and $350\,\mathrm{kg\,m^{-3}}$ respectively. The portion of the concrete specimen under uniform tension has a volume of $160\times 100\times 70\,\mathrm{mm}^3$. All specimens were cured in water for 8 to 9 weeks at 21° C. The frequency used for the cyclic tests was 240 c.p.m.

The results of the direct tensile strength of fast fracture are shown in Fig. 2. It is found that there is substantially no difference in the tensile strengths between specimens cured for 8 and 9 weeks. Thus all results are used in Fig. 2 to get a better estimation of the Weibull parameters. It is found that

$$\ln\ln\frac{1}{1-F} = 20.27\ln\sigma - 24.833\ (\mathrm{MPa}) \quad (20)$$

Thus, the Weibull modulus for this plain concrete is 20.27.

The cyclic stress used by Saito and Imai [21] has the form

$$\sigma(t) = 1.666[(S+0.08) + (S-0.08)\sin(2\pi ft)]\ (\mathrm{MPa}) \quad (21)$$

where $S = \sigma_{\max}/\sigma_{0.5}$ and $\sigma_{0.5}$ is given by Equation 20 with $F = 50\%$. The values of S used are 0.75, 0.775,

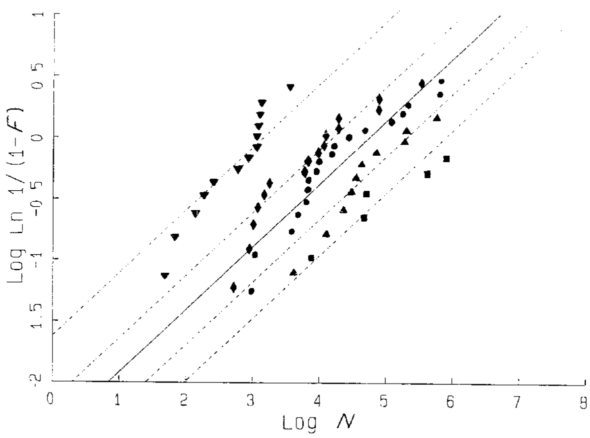

Figure 3 Cyclic fatigue of plain concrete after Saito and Imai [21]; (——) best fit; (– – –) predictions; S = (▼) 0.875; (◆) 0.825; (●) 0.8; (▲) 0.775; (■) 0.75.

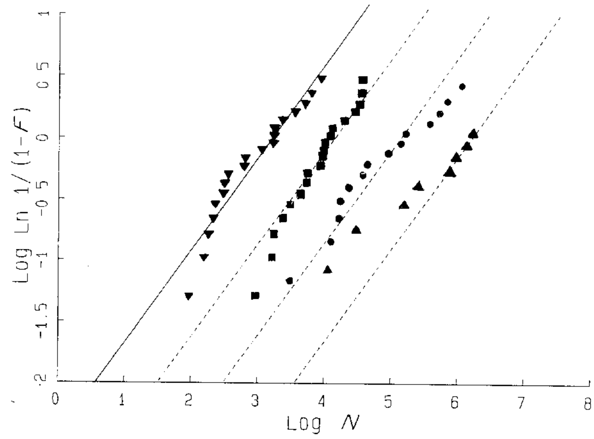

Figure 4 Cyclic fatigue of lightweight concrete after Saito [22]; (——) best fit; (– – –) predictions; S = (▼) 0.919; (■) 0.871;(●) 823; (▲) 0.774.

0.8, 0.825, 0.85 and 0.875 (or $0.807 < \zeta < 0.833$). Equation 16 is applied to the data of $S = 0.8$ because the maximum number of specimens were tested at this stress condition. It is found that

$$\log \ln \frac{1}{1 - F} = 0.515 \log N - 2.435 \quad (22)$$

so that $m/(n - 2) = 0.515$. Because $m = 20.27$, we have $n = 41.36$ which is the stress corrosion exponent of Equation 5.

Predictions of lifetimes, N, according to Equation 16 for stress conditions of $S = 0.75$, 0.775, 0.825 and 0.875 are shown in Fig. 3 together with the experimental results. The agreement is not unreasonable.

Similar tensile fatigue experiments on lightweight concrete were also performed by Saito [22]. Dimensions of samples and curing conditions were kept the same as those of plain concrete [21]. The coarse aggregate was a round type with 15 mm maximum size. The ratio of the minimum dimension of the tensile specimens to the maximum size of coarse aggregate was 4.67. The water–cement ratio, sand–cement ratio and cement content were maintained at 0.50, 0.45 and 348 kg m^{-3}, respectively. Again the frequency used for the cyclic stress was 240 c.p.m. Values of S used in Equation 21 were 0.919, 0.871, 0.823 and 0.774, respectively (or $0.812 < \zeta < 0.840$).

Fatigue results of lightweight concrete are shown in Fig. 4. Applying Equation 16 to the data of S = 0.919, we find that

$$\log \ln \frac{1}{1 - F} = 0.758 \log N - 2.435 \quad (23)$$

Unfortunately, only the average tensile strength of fast fracture was provided so the Weibull modulus cannot be worked out in terms of Equation 1. But if the same value of n as that of the plain concrete is assumed for the lightweight concrete (so that m is now determined), predictions from Equation 16 agree very well with the fatigue data. Alternatively, two sets of data for two different S can be used to fit Equation 16 and the values of m and n solved simultaneously. This gives n approximately equal to 40 which is close to the assumed value 41.36.

4. Discussion and conclusion

The reasonably good agreement between the theoretically predicted lifetimes and experimental results suggests that the fracture statistics together with the simple crack growth model can be applied to both plain and lightweight concrete despite their highly inhomogeneous structures. However, it should be pointed out that the successful application of this theory originally developed for single-phase homogeneous materials to heterogeneous concretes has its empirical feature because cracks in these latter materials are most likely to be stabilized by sands and aggregates. Thus the Weibull parameters obtained through Equations 1 to 4 can only be taken as statistically averaged flaw characteristics. The same argument holds for the parameters (A, n) in the crack growth law of Equation 5.

Because the same theory gives reasonable predictions for both plain and lightweight concrete, and the stress corrosion exponents, n, are the same for both materials, it can be concluded that the fatigue mechanism is identical in both materials. This conclusion can only be true if the slow crack growth is mainly confined within the cement matrix or the matrix–aggregate boundaries. The similar matrix constituents used in both materials seem to support this argument. Unlike Tait [15], we cannot conclude that the time-dependent fracture of the plain and lightweight concrete under cyclic stress conditions is purely due to the environmentally assisted slow crack growth indicated by Equation 5. To do so we need to be able to predict the cyclic fatigue results from either constant stress rate or constant sustained stress data. This is done by Tait for mortar [15] meaning that cyclic fatigue-induced damage is negligible.

For concrete the picture is not clear; but there is evidence that cyclic fatigue-induced damage has occurred in plain concrete beams [23] subjected to pure bending (see Fig. 5). For a given maximum stress, Equation 19 shows that the smaller the stress ratio $R(=\sigma_{\min}/\sigma_{\max})$ the longer the fatigue life. However, Fig. 5 shows the opposite trend. In this case Equation 5 needs to be modified to

$$\frac{da}{dt} = AK^n + fA_c \Delta K^{n_c} \quad (24)$$

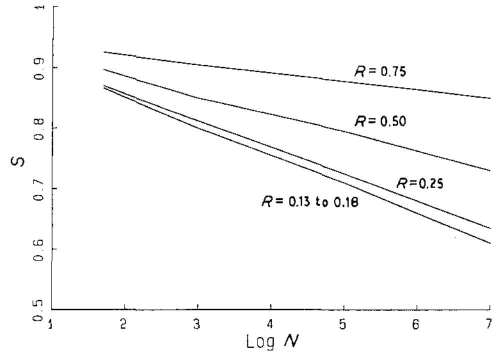

Figure 5 Effect of the range of stress on the behaviour of plain concrete under bending fatigue after Murdock and Kesler [23].

where the first term on the right-hand side is the crack velocity component due to environmentally assisted crack growth and the second term is that due to the cyclic fatigue-induced damage. If σ_{min} (< 0) is fixed in a cyclic fatigue, both mechanisms will increase the crack growth velocity with increasing σ_{max}. But if σ_{max} is fixed and σ_{min} is reduced, the effect of the cyclic fatigue-induced damage will be enhanced while the effect of the environmentally assisted slow crack growth will be reduced. Whether the resultant crack growth velocity increases or decreases will depend on which mechanism has more influence than the other. This argument may explain why the tensile fatigue data [21, 22] (fixed σ_{min}) agree with our present theory, while the bending [23] fatigue data (changing σ_{min}) do not. Also, it can be seen from Fig. 1 that G_* is essentially a constant for $\zeta > 0.087 < R < 0.107$. Under these conditions, Equations 16 and 19 both show that the fatigue lifetime only depends on σ_{max}. In Fig. 5 it can be seen that the bending fatigue life is independent of R if $R < 0.2$ (noting $S = \sigma_{max}/\sigma_{0.5}$ and $\sigma_{0.5}$ = const.) and this is in agreement with the fact that G_* is a constant. For $R \geqslant 0.25$ the lifetime N increases with R for any given S showing that the fatigue-induced damage effect is reduced. Nevertheless such an effect does exist and Equations 16 and 19 are incapable of accurate lifetime predictions.

If the fatigue life N only depends on σ_{max} (such as at low R), no matter whether one or both crack growth mechanisms exist, it is possible to find equivalent A^* and n^* values to recast Equation 24 in the form of Equation 5. (At low R ratios, Equation 14 can be reduced to: $\dot{a} = A^* K_{max}^{n^*}$ because the second term on the right-hand side is largely determined by σ_{max} and hence K_{max}. At large R-ratios, however, this second term depends on R and not σ_{max} (or K_{max}) alone.)The tensile fatigue data for concrete in Figs 3 and 4 are for $R = 0.1$ and that both crack growth mechanisms are likely to have occurred. Thus, n ($\simeq 40$) determined from these data is, in fact, the equivalent n^*. Lifetime predictions can therefore be made for different S using this value of n^* as shown in these figures.

The effect of cyclic-induced damage has also been found in uniaxial tensile fatigue of concrete by Cornelissen and Reinhardt [24] although the effect is not as clear in reversed bending fatigue by McCall [25]. Thus fatigue life predictions exclusively based on the environmentally assisted slow crack growth for concrete under cyclic stresses may have problems, especially if a large fracture process zone is developed ahead of the crack, or if asperity contacts along the crack surfaces exist [26]. It is concluded, therefore, that in general both slow crack growth mechanisms due to environmental and cyclic fatigue effects need to be considered in the fracture of concrete subjected to cyclic loading. The simple statistical fracture theory applies only for low R-ratios, provided the equivalent fracture parameters m, n, A, C_B or C_T can be found.

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